You can answer in any order that you like, but indicate the question numbers in your answers. The questions are roughly sorted by difficulty except for ★ questions, which are somewhat harder.

1 Universal Languages and Time Bounds

Place in as low a complexity class as you can.

(a) $L := \{(M, x) : M(x) \text{ halts}\}$
(b) $L := \{M : \exists x\ M(x) \text{ halts}\}$
(c) $L := \{(M, x, t) : M(x) \text{ halts within time } t\}$
(d) $L := \{(M, t) : \exists x\ M(x) \text{ halts within time } t\}$
(e) $L := \{(M, t) : \forall x\ M(x) \text{ halts within time } t\}$
(f) $L := \{(M, 1^t) : \forall x\ M(x) \text{ halts within time } t\}$

Which languages are complete for their classes? Briefly justify your answers.

2 Relations between Classes

Assume **P** = **PSPACE**. Now consider each of the following statements and say whether it is true, false, or an open question. In case you say true or false, also give a short justification of your answer.

(a) **P** = **NP**
(b) **NP** = **coNP**
(c) **P** = **L**
(d) **NP** = **EXP**
(e) **NP** = **L**
(f) **NPSPACE** = **P**

3 Kleene Star

The Kleene star of a language L is the language

$$L^\star := \{x_1 \cdots x_k : k \geq 0 \text{ and } x_1, \ldots, x_k \in L\} .$$

That is, $L^\star$ consists of strings formed by concatenating a finite number of elements of L.

(a) Show that **NP** is closed under Kleene star.
(b) Show that **P** is closed under Kleene star.

4 PCP Variants

Determine which of the following variants of PCP are decidable:

(a) PCP over a unary alphabet.
(b) PCP over a binary alphabet.
★(c) PCP (over an arbitrary alphabet, as usual) but now the goal is determine whether or not there is an *infinite* sequence $i_1, i_2, \ldots$ such that $t_{i_1} t_{i_2} \cdots = b_{i_1} b_{i_2} \cdots$, where $\{(t_i, b_i)\}_{i=1}^n$ is a finite set of tiles.

5 Randomization and Nondeterminism

A language $L \in \mathbf{BP} \cdot \mathbf{NP}$ if there exists a polynomial-time deterministic Turing machine M such that

$$x \in L \implies \Pr_{r \in \{0,1\}^{m(n)}} \left[\exists y \in \{0,1\}^{k(n)}\ M(x,y;r) = 1\right] \geq 2/3$$
$$x \notin L \implies \Pr_{r \in \{0,1\}^{m(n)}} \left[\exists y \in \{0,1\}^{k(n)}\ M(x,y;r) = 1\right] \leq 1/3$$

where $m(n), k(n) \leq \text{poly}(n)$. A language $L \in \mathbf{NP} \cdot \mathbf{BP}$ if there exists a polynomial-time deterministic Turing machine M such that

$$x \in L \implies \exists y \in \{0,1\}^{k(n)} \Pr_{r \in \{0,1\}^{m(n)}} [M(x,y;r) = 1] \geq 2/3$$
$$x \notin L \implies \forall y \in \{0,1\}^{k(n)} \Pr_{r \in \{0,1\}^{m(n)}} [M(x,y;r) = 1] \leq 1/3$$

where $m(n), k(n) \leq \text{poly}(n)$. Show that

$$\mathbf{NP} \cdot \mathbf{BP} \subseteq \mathbf{BP} \cdot \mathbf{NP} .$$

6 Separation via Closure

For a language $L \subseteq \Sigma^\star$ and a function $f : \Sigma^\star \longrightarrow \Sigma^\star$, let

$$L_f := \{x \in \Sigma^\star : f(x) \in L\} .$$

We say that a complexity class is *closed under (polynomial) composition* if for any L in the class and any polynomial-time computable function f it is the case that language L_f is also in that class.

For a language A, let $A^\sharp := \{x0^{|x|^2 - |x|} : x \in A\}$ be the language consisting of elements x of A appended with $|x|^2 - |x|$ zeros.

(a) Show that $\mathbf{NP}$ is closed under composition.

(b) Show that if $A \in \mathbf{SPACE}(n^2)$ then $A^\sharp \in \mathbf{SPACE}(n)$.

(c) Show that $\mathbf{SPACE}(n)$ is *not* closed under composition.

(d) Conclude that $\mathbf{NP} \neq \mathbf{SPACE}(n)$.

7 NP-Completeness

CLIQUE-COVER is the following problem:

INPUT: A graph $G = (V, E)$ and a positive integer $K \leq |V|$;

QUESTION: Can the vertices of G be partionated into $k \leq K$ disjoint sets $V_1, \ldots, V_k$ such that for every $i = 1, \ldots, k$ the subgraph induced by V_i is a complete graph?

Show that CLIQUE-COVER is **NP**-complete.

8 Bonus Question: Sometimes Space = Time

Show that there exists a function $T(n) \geq n$ such that $\mathbf{DTIME}(T(n)) = \mathbf{DSPACE}(T(n))$.